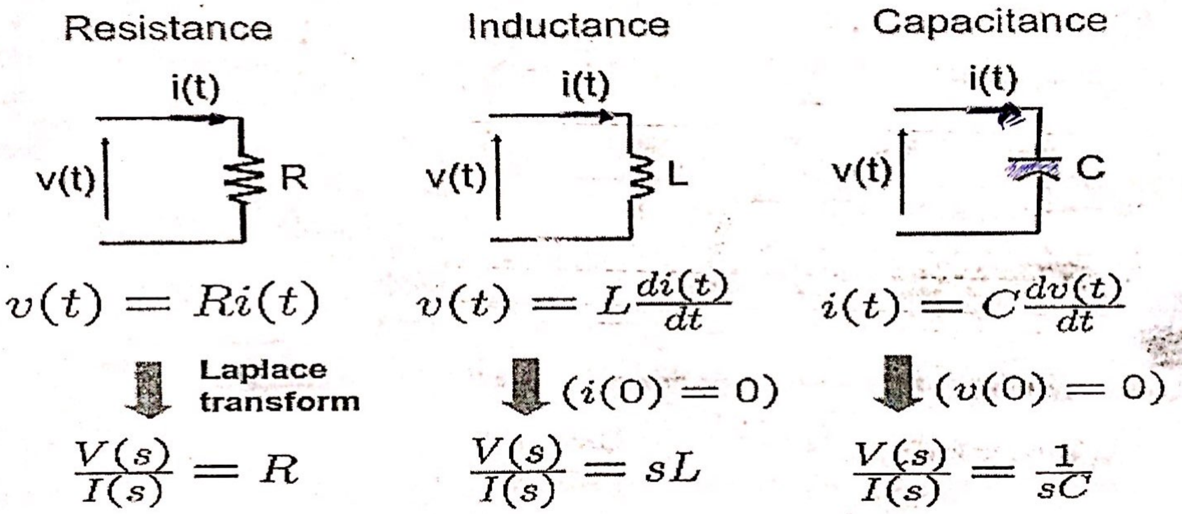
Week #3

Transfer Function ……………………………………EE362

1- Modeling Electrical Networks

- Electrical elements: ***resistors, capacitors, inductors, amplifiers***
- Combining electrical elements: ***networks***
- Transfer functions of electrical elements = ***output/ input***.

Models of electrical elements:-

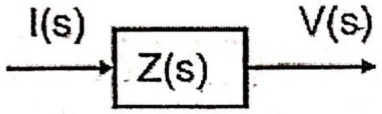


$v(t) = Ri(t)$ $\quad$ $v(t) = L\frac{di(t)}{dt}$ $\quad$ $i(t) = C\frac{dv(t)}{dt}$

$(i(0) = 0)$ $\quad$ $(v(0) = 0)$

$\frac{V(s)}{I(s)} = R$ $\quad$ $\frac{V(s)}{I(s)} = sL$ $\quad$ $\frac{V(s)}{I(s)} = \frac{1}{sC}$

Z(s): V(s)=Z(s)I(s)



Element	Time domain	Impedance Z(s)
Resistance	$v(t) = Ri(t)$	$\frac{V(s)}{I(s)} = \boxed{R}$
Inductance	$v(t) = L\frac{di(t)}{dt}$	$\frac{V(s)}{I(s)} = \boxed{sL}$
Capacitance	$i(t) = C\frac{dv(t)}{dt}$	$\frac{V(s)}{I(s)} = \boxed{\frac{1}{sC}}$

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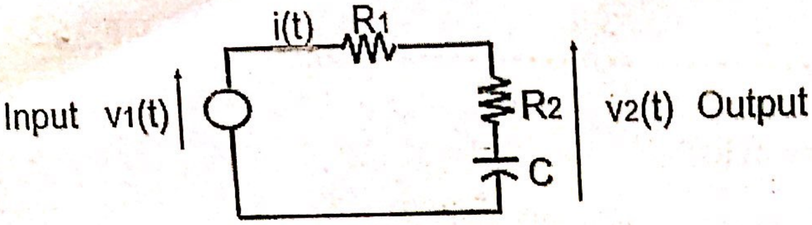
Modeling Example



- Kirchhoff voltage law (with zero initial conditions)

$$\begin{aligned} v_1(t) &= (R_1+R_2)i(t) + \tfrac{1}{C}\int_0^t i(\tau)d\tau \\ v_2(t) &= R_2 i(t) + \tfrac{1}{C}\int_0^t i(\tau)d\tau \end{aligned}$$

- By Laplace transform,

$$\begin{aligned} V_1(s) &= (R_1+R_2)I(s) + \tfrac{1}{sC}I(s) \\ V_2(s) &= R_2 I(s) + \tfrac{1}{sC}I(s) \end{aligned}$$

- Transfer function

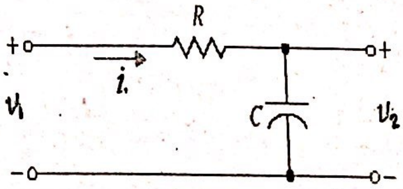
$$\begin{aligned} G(s) = \frac{V_2(s)}{V_1(s)} &= \frac{R_2+\frac{1}{sC}}{(R_1+R_2)+\frac{1}{sC}} \\ &= \frac{R_2Cs+1}{(R_1+R_2)Cs+1} \end{aligned}$$

(first-order system)

Ex: Find the transfer function relating voltage ,Vc(s), to the input voltage V(s).

$$V_1(s) = \left(R + \frac{1}{Cs}\right)\cdot I(s) \qquad Z_1(s) = R$$

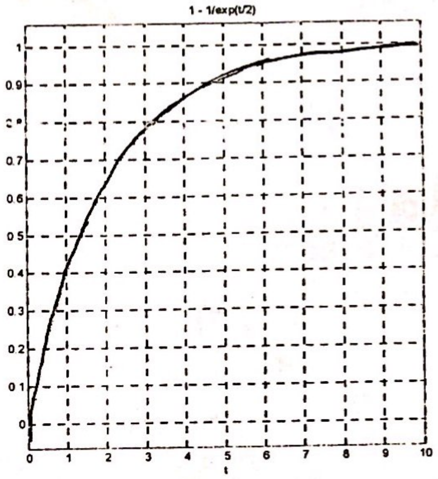
$$V_2(s) = \left(\frac{1}{Cs}\right)\cdot I(s) \qquad Z_2(s) = \frac{1}{Cs}$$



$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{1}{RCs+1}$$

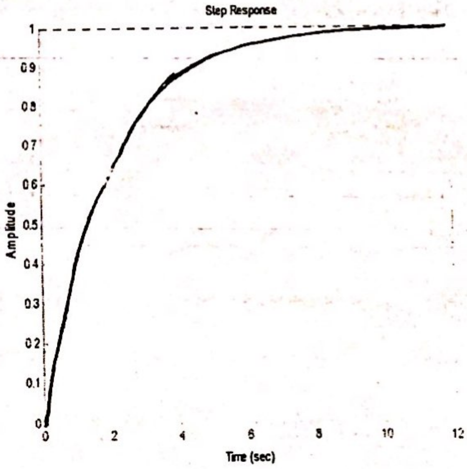
MATLAB Code

```
syms s
C= 1 ;                 % c=1 micro farad
R = 2     ;            % R= 2 kilo ohm
vi = 1       ;         % vi(t) =1 volt(step function)
Vi   = 1/s ;           % Vi(s) = 1/s
Vc= (1/(R*C*s + 1))*Vi;
disp('Vc(s)')
pretty(Vc)
vc= ilaplace(Vc);
disp('vc(t)')
pretty(vc)
ezplot(vc,[0 10])
grid
```



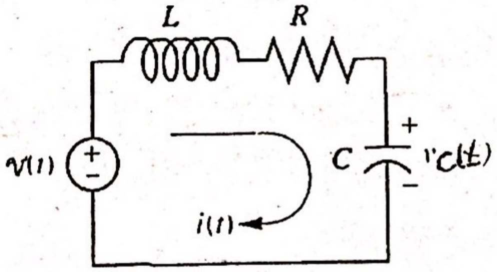
MATLAB Code

```
C= 1;
R=2;
num=1;
den= [R*C  1];
G= tf(num,den)
step(G)
grid
xlabel('time'
ylabel(' Amp. of Vc')
```



Ex: Find the transfer function relating the capacitor voltage, Vc(s), to the input voltage, $V_i(s)$

Summing the voltages around the loop.



RLC network

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$LS\ I(s) + R.I(s) + I(s)/SC = V(s)$$

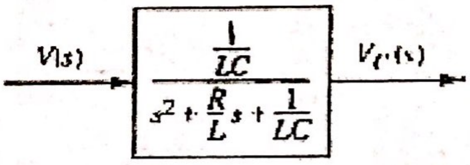
Input : V(s)

Output : $Vc(s) = I(s)\ .Zc(s) = I(s).(1/sc)$ $\xrightarrow{yields}$ $I(s) = sc\ Vc(s)$

$$(LCs^2 + RCs + 1)V_C(s) = V(s)$$

Solving for the transfer function, Vc(s)/V(s), we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



$$\frac{Vc(s)}{V(s)} = \frac{1}{S^2LC + SCR + 1}$$

Another solution using voltage division and the transformed circuit.

$$Vc(s) = I(s)\ .\ Zc(s)$$

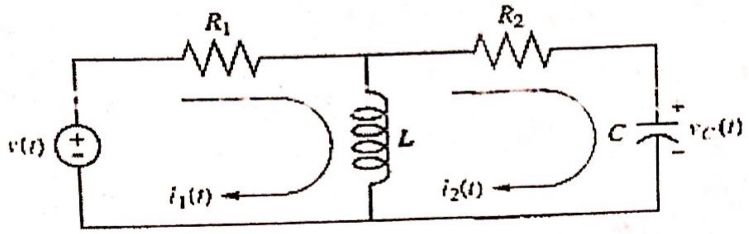
$$Vc(s) = \frac{V(s)}{SL + R + \frac{1}{SC}} \cdot \frac{1}{Sc} \xrightarrow{yields} Vc(s) = \frac{V(s)}{S^2LC + SCR + 1} \xrightarrow{yields} \frac{Vc(s)}{V(s)} = \frac{1}{S^2LC + SCR + 1}$$

Complex Circuits via Mesh Analysis

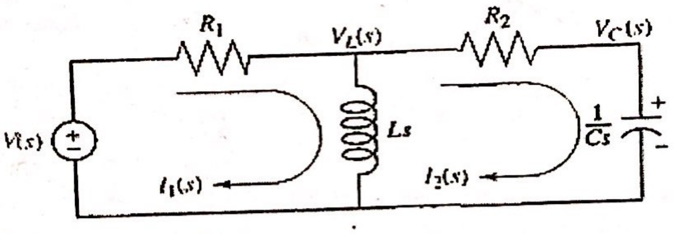
To solve complex electrical networks-those with multiple loops and nodes-using mesh analysis, we can perform the following steps:

1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh.
4. Write Kirchhoffs voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Ex. Find the transfer function, $G(s) = I_2(s)/V(s)$, for the following network.



The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.



Around Mesh 1, where $I_1(s)$ flows,

$$R_1I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

Around Mesh 2, where $I_2(s)$ flows,

$$LsI_2(s) + R_2I_2(s) + \frac{1}{Cs}I_2(s) - LsI_1(s) = 0$$

Combining terms of two equations,

$$\begin{aligned}(R_1 + Ls)I_1(s) \qquad\qquad &- LsI_2(s) = V(s)\\ -LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)&I_2(s) = 0\end{aligned}$$

We can use **Cramer's rule** (or any other method for solving simultaneous equations) to solve the equations for $I_2(s)$.

$$I_2(s) = \frac{\begin{vmatrix}(R_1 + Ls) & V(s)\\ -Ls & 0\end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta} \qquad \text{where} \qquad \Delta = \begin{vmatrix}(R_1 + Ls) & -Ls\\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right)\end{vmatrix}$$

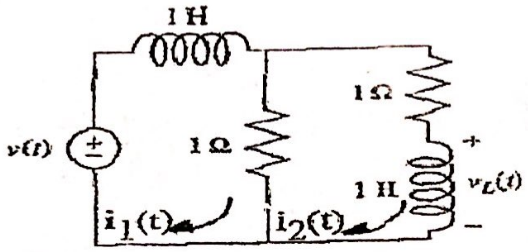
Forming the transfer function, $G(s)$, yields

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

* tfdata

* numden → ~~Poly~~ Sym2Poly

Ex. Find the transfer function relating the inductor voltage, $V_L(s)$, to the input voltage, $Vi(s)$.



Writing mesh equations $$(s+1)I_1(s) - I_2(s) = V_i(s)$$

$$-I_1(s) + (s+2)I_2(s) = 0$$

But, $I_1(s) = (s+2)I_2(s)$. Substituting this in the first equation yields,

$$(s+1)(s+2)I_2(s) - I_2(s) = V_i(s)$$

or

$$I_2(s)/V_i(s) = 1/(s^2 + 3s + 1)$$

But, $V_L(s) = sI_2(s)$. Therefore, $V_L(s)/V_i(s) = s/(s^2 + 3s + 1)$.

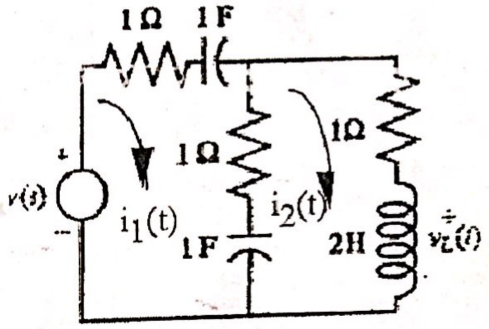
MATLAB Code

```
clc
syms s V
a=[ s+1 V ; -1 0];
b=[ s+1 -1 ; -1 s+2];
I2= det(a)/det(b);
disp('I2(s)')
pretty(I2)
L=1 ;           % L=1 H
Vo = I2*s*L ;  % V0=VL= sL*I2 =s*I2
G=Vo/V;
sprintf('\n')
disp('G(s) = ')
pretty(G)
```

```
                    V
I2(s) =     ------------
             2
            s  + 3 s + 1


                     s
G(s)  =     ------------
             2
            s  + 3 s + 1
```

Ex. Find the transfer function relating the inductor voltage, $V_L(s)$, to the input voltage, $V_i(s)$



$$(2+\frac{2}{s})I_1(s)-(1+\frac{1}{s})I_2(s)=V(s)$$

$$-(1+\frac{1}{s})I_1(s)+(2+\frac{1}{s}+2s)I_2(s)=0$$

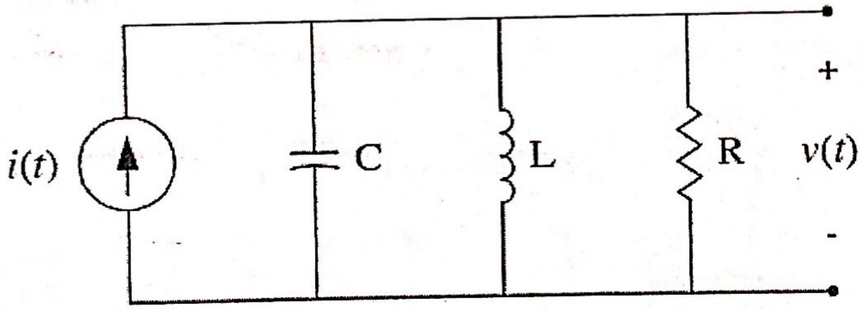
Solving for $I_2(s)$:

$$I_2(s)=\frac{\begin{vmatrix}\frac{2(s+1)}{s} & V(s)\\ -\frac{s+1}{s} & 0\end{vmatrix}}{\begin{vmatrix}\frac{2(s+1)}{s} & -\frac{s+1}{s}\\ -\frac{s+1}{s} & \frac{2s^2+2s+1}{s}\end{vmatrix}}=\frac{V(s)s}{4s^2+3s+1}$$

Therefore,

$$\frac{V_L(s)}{V(s)}=2s\frac{I_2(s)}{V(s)}=\frac{2s^2}{4s^2+3s+1}$$

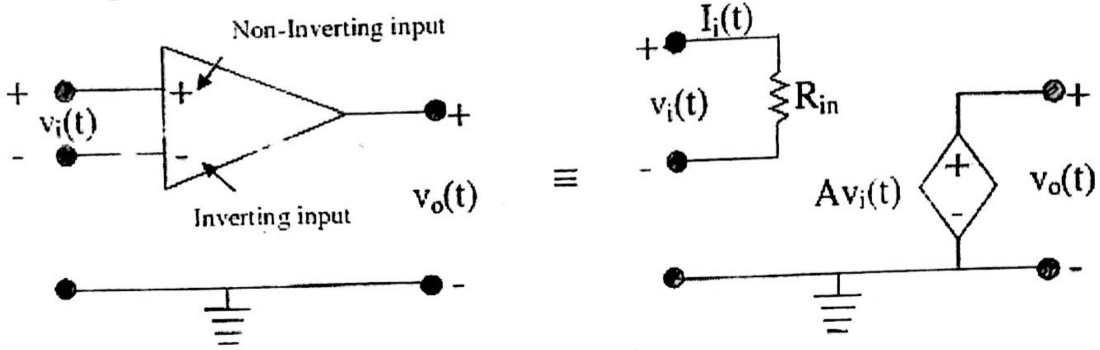
Ex. f or the circuit shown below write the transfer function $G(s) = V(s)/I(s)$



$$I(s)=\frac{V(s)}{1/_{sC}}+\frac{V(s)}{sL}+\frac{V(s)}{R}$$

$$I(s)=V(s)\left[\frac{1}{1/_{sC}}+\frac{1}{sL}+\frac{1}{R}\right]\rightarrow\frac{V(s)}{I(s)}=\frac{1}{\left[\frac{1}{1/_{sC}}+\frac{1}{sL}+\frac{1}{R}\right]}$$

Operation Amplifiers



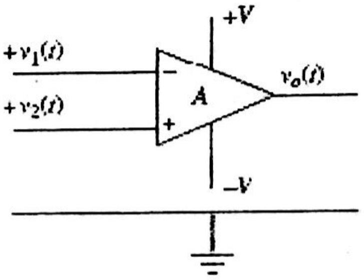
A general op-amp model is described above. In practice, the input resistance, R_{in} is very large ($> 10^{12}$ ohm), and the gain, A, is very large ($> 10^5$). Thus, we will use the ideal model in the analysis:

1. Input current Ii(t) = 0 (due to the large input impedance)

2. Input voltage difference $v_i(t) = 0$.

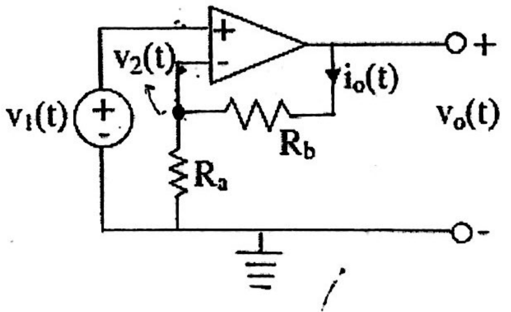
An operation amplifier has the following characteristics:

1. Differential input, v2(t)-v1(t)
2. High input impedance, Zi = ∞ (ideal)
3. Low output impedance, Zo = 0 (ideal)
4. High constant gain amplification, A= ∞ (ideal)



The output, vo(t), is given by **vo(t) = A(v2(t) – v1(t))**

Ex Find vo(t) / vi(t)



$U_2 = U_1 = U_i$

$\frac{U_o}{R_a + R_b} * R_a = U_i$

$\frac{U_o}{U_i} = \frac{R_a + R_b}{R_a}$

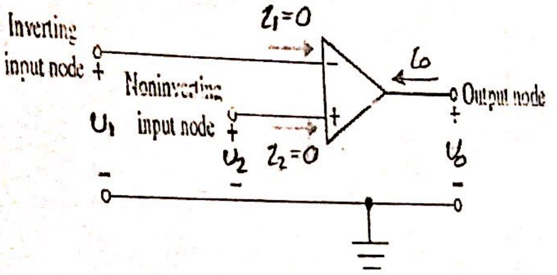
Based on the ideal op-amp model,

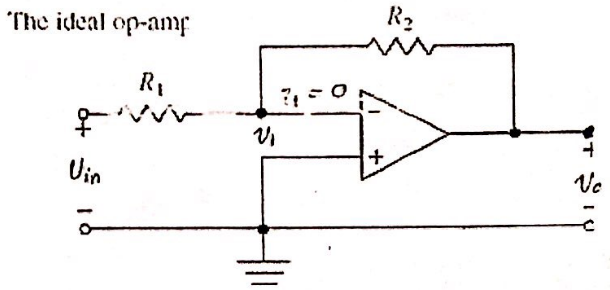
$$v_2(t) = v_1(t)$$

Also, as the op-amp does not have any input current, applying KCL at the inverting port, we have

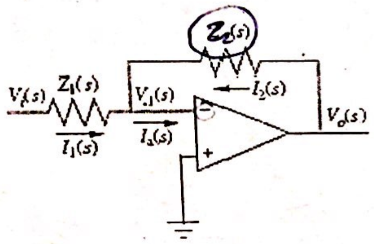
$$v_2(t)/R_a = (v_o(t) - v_2(t))/R_b$$
$$v_o(t)/v_2(t) = 1 + R_b/R_a$$
$$v_o(t)/v_1(t) = 1 + R_b/R_a \Rightarrow \frac{V_o(t)}{V_i(t)} = \frac{R_a + R_b}{R_a}$$

This circuit is called *Non-Inverting Amplifier.*





Inverting Operational amplifiers



$V1 = 0$, $Ia = 0$

$I1 + I2 = 0$ $(I1=-I2)$

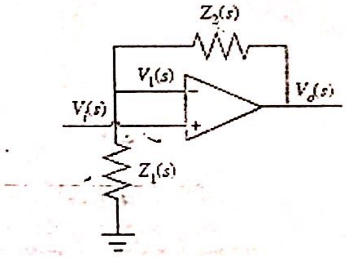
$Vi - V1 = Vi - 0 = I1 \cdot Z1$

$Vo - V1 = Vo - 0 = I2 \cdot Z2$

Combining, we obtain ,

$$\frac{V_0(s)}{V_i(s)} = -\frac{Z_{2(s)}}{Z_1(s)}$$

Noninverting Operational Amplifiers



$V0(s) = A(Vi(s) - V1(s))$ ……(a)

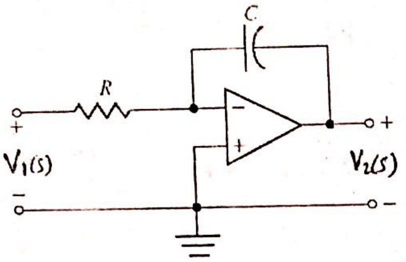
But, using voltage division,

$$V1(s) = \frac{Z1(s)}{Z1(s)+z2(s)}\, Vo(s) \quad ……(b)$$

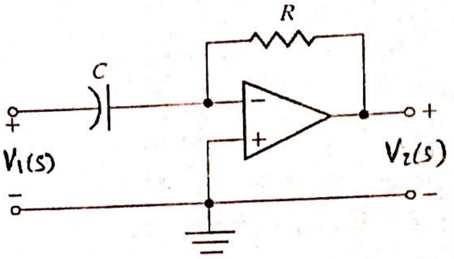
Substituting eq(b) in eq(a)

$$\frac{V0(s)}{Vi(s)} = \frac{A}{1+\frac{AZ1(s)}{(Z1(s)+Z2(s))}}$$,for large A

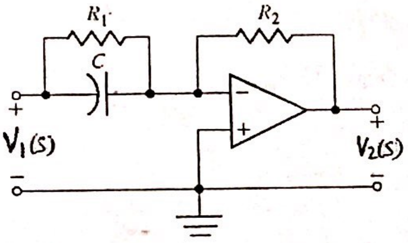
$$\frac{V_0(s)}{V_i(s)} = \frac{Z_1(s)+Z_{2(s)}}{Z_1(s)}$$



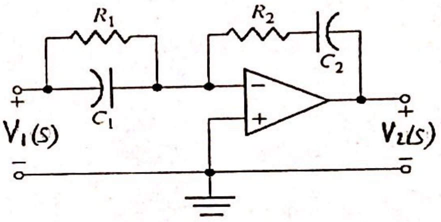
$$\frac{V_2(s)}{V_1(s)} = \frac{-1}{RCs}$$



$$\frac{V_2(s)}{V_1(s)} = -RCs$$

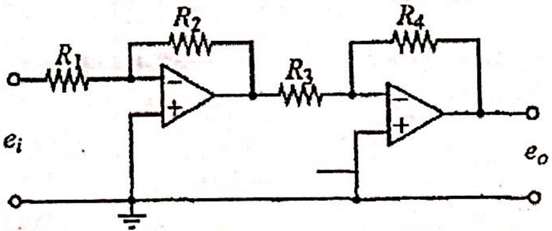


$$\frac{V_2(s)}{V_1(s)} = \frac{R_2(R_1 \cdot C \cdot s + 1)}{R_1}$$

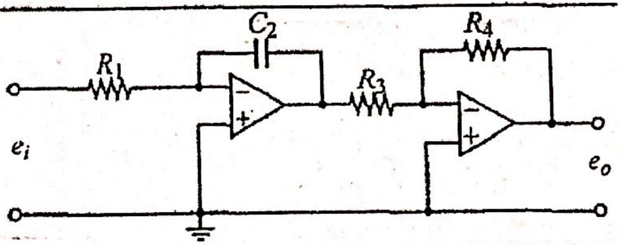


$$\frac{V_2(s)}{V_1(s)} = \frac{-(R_1 \cdot C_1 \cdot s + 1)(R_2 \cdot C_2 \cdot s + 1)}{R_1 \cdot C_2 \cdot s}$$

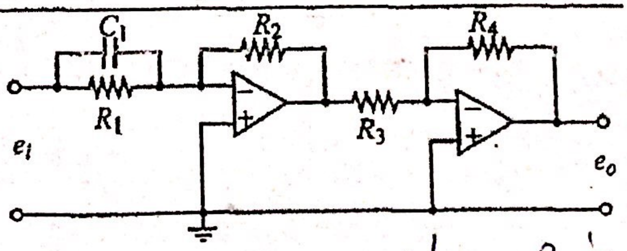
$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{R_4}{R_3}\frac{R_2}{R_1}$$



$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{R_4}{R_3}\frac{1}{R_1 C_2 s}$$

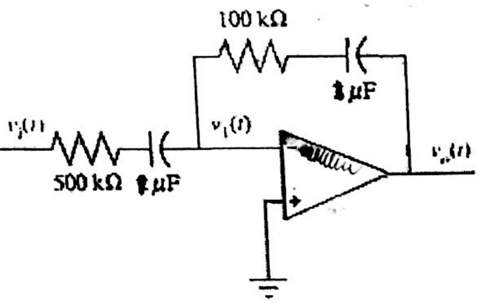


$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{R_4}{R_3}\frac{R_2}{R_1}(R_1 C_1 s + 1)$$



$R_1 \, // \, \frac{1}{Cs}$ $\quad \frac{R_1 \frac{1}{Cs}}{R_1 + \frac{1}{Cs}}$

Ex. Find the transfer function, G(s) =V0(s)/Vi(s) for each of the operational amplifiers circuits shown in figure below.

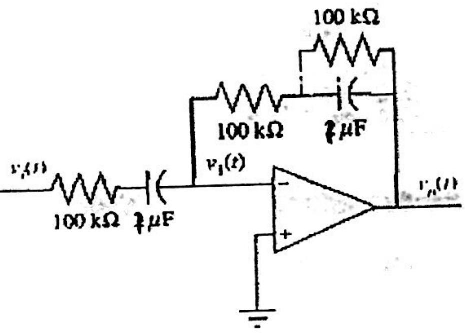


$$Z_1(s) = 5x10^5 + \frac{10^6}{s}$$

$$Z_2(s) = 10^5 + \frac{10^6}{s}$$

Therefore,

$$-\frac{Z_2(s)}{Z_1(s)} = -\frac{1}{5}\frac{s+10}{s+2}$$

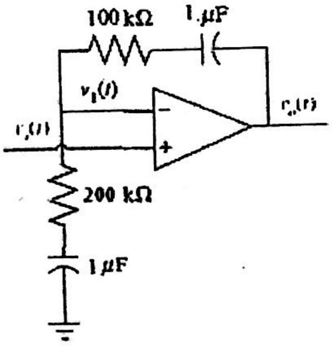


$$Z_1(s) = 100000 + \frac{1}{1\times10^{-6}s} = 100000\frac{s+10}{s}$$

$$Z_2(s) = 100000 + \frac{1}{1\times10^{-6}s + \frac{1}{100000}} = 100000\frac{s+20}{s+10}$$

Therefore,

$$G(s) = -\frac{Z_2}{Z_1} = -\frac{(s+20)s}{(s+10)^2}$$



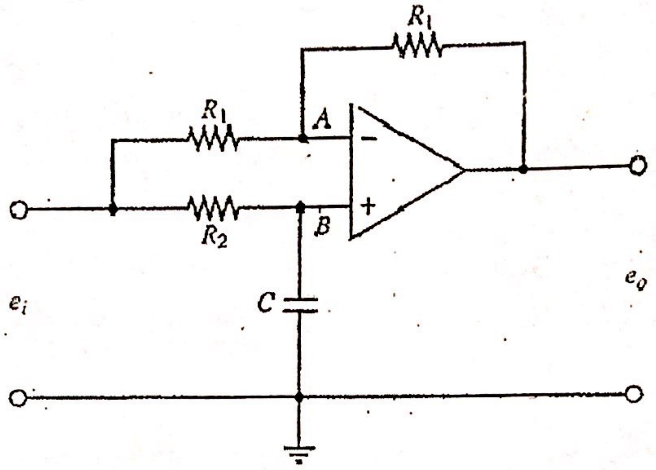
$$Z_1(s) = 200000 \div \frac{1}{1\times10^{-6}s}$$

$$Z_2(s) = 100000 + \frac{1}{1\times10^{-6}s}$$

Therefore,

$$G(s) = \frac{Z_1+Z_2}{Z_1} = \frac{3}{2}\frac{s+\frac{20}{3}}{s\div5}.$$

Ex. Obtain the transfer function *Eo(s)/Ei(s)* of the op-amp circuit shown in Figure bellow.



The voltage at point A is

$$e_A = \frac{(e_i - e_o) \cdot R_1}{2R_1} + e_o$$

$$e_A = \frac{1}{2}(e_i - e_o) + e_o \qquad e_A = \frac{1}{2}(e_i - e_o) + e_o$$

The Laplace transform of this last equation is

$$E_A(s) = \frac{1}{2}\left[E_i(s) + E_o(s)\right]$$

The voltage at point B is

$$E_B(s) = \frac{\frac{1}{Cs}}{R_2 + \frac{1}{Cs}} E_i(s) = \frac{1}{R_2Cs + 1} E_i(s)$$

Since $[E_B(s) - E_A(s)]\,A = E_o(s)$ and $A \gg 1$, we must have $E_A(s) = E_B(s)$ Thus

$$\frac{1}{2}\left[E_i(s) + E_o(s)\right] = \frac{1}{R_2Cs + 1} E_i(s)$$

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2Cs - 1}{R_2Cs + 1} = -\frac{s - \frac{1}{R_2C}}{s + \frac{1}{R_2C}}$$

Modeling Mechanical System

Models of mechanical systems are important in control engineering because a mechanical system may be a vehicle, a robotic arm, a missile, or any other system which incorporates a mechanical component. Mechanical systems can be divided into two categories: translational systems and rotational systems. Some systems may be either purely translational or purely rotational, whereas others may be hybrid, incorporating both translational and rotational components.

Modeling mechanical elements by ordinary differential equations (ODEs).

(1) –**Translation**
- mass, damper, spring

(2) – **Rotation**
- inertia, rotary damper, rotary spring, gear

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Translational Mechanical Systems

The basic building blocks of translational mechanical systems are **mass, spring, and damper**. The input to a translational mechanical system may be a force F and the output the displacement x.

Spring : For a linear spring, the extension **x** is proportional to the applied force **F** as given in equation

$$\mathbf{F=K\,.\,x}$$

where k is known as the stiffness constant. K = N/m (Newtons/meter)

Damper : damper element is a form of damping and can be considered to be represented by a piston moving in a viscous medium in a cylinder. As the piston moves the liquid passes through the edges of the piston, damping the movement of the piston. The force **F** which moves the piston is proportional to the velocity of the piston movement and is given by

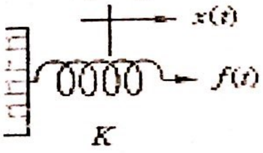
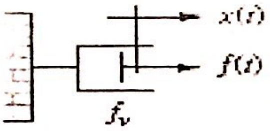
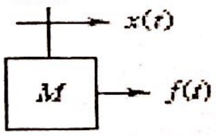
$$F = f_v \frac{dx}{dt}$$

where fv is known as the coefficient of viscous friction. fv = N-s/m (Newtons-second/meter)

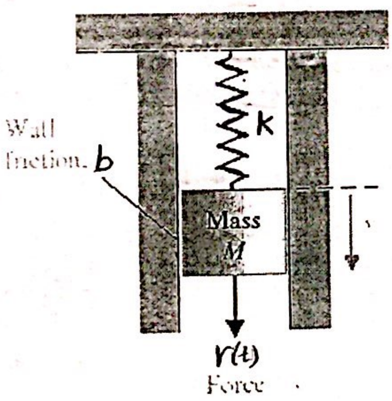
Mass: When a force is applied to a mass, the relationship between the force F and the acceleration of the mass is given by Newton's second law as F = Ma. Since acceleration is the rate of change of velocity and the velocity is the rate of change of displacement,

$$\mathbf{F = M\frac{d^2x}{dt^2}}$$

Where M: mass (kg)

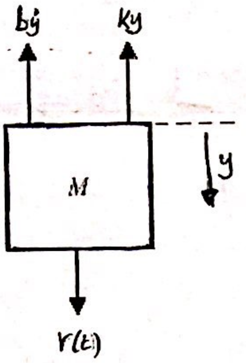
Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring  	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	$\boxed{K}$
Viscous damper  	const given $f(t) = f_v v(t)$	$f_v S X(s)$ $f(t) = f_v \frac{dx(t)}{dt}$ $\longrightarrow$ Sterm	$\boxed{f_v s}$
Mass  	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	$\boxed{Ms^2}$

[some of impedance]X(s) = [sum of applied force]



(a)

Spring- mass-damper system



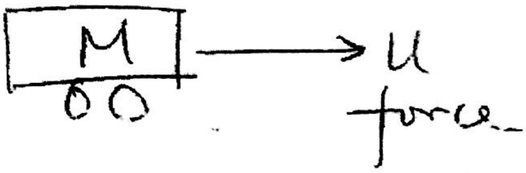
(b)

Free- body diagram

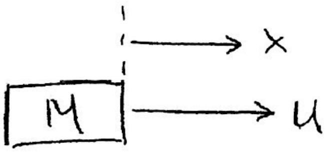
$$M\cdot\frac{d^2}{dt^2}y(t) + b\cdot\frac{d}{dt}y(t) + k\cdot y(t) = r(t)$$

2

Ex) One-mass system



Neglect rotational inertia of wheels, friction, etc.,



freebody diagram

Apply Newton's Law

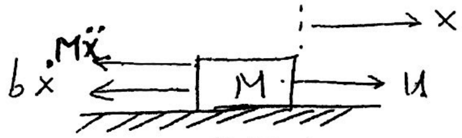
$$u = M\ddot{x}$$

Take Laplace transform

$$u(s) = M \cdot s^2 X(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{Ms^2}$$

Ex) one-mass system with friction



we model friction as proportional to velocity.

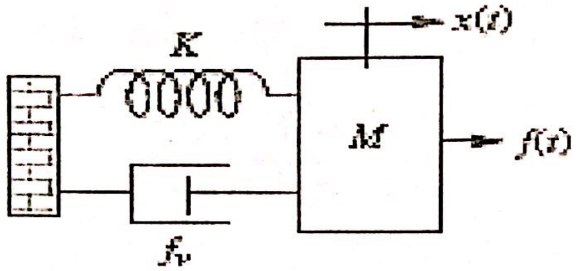
$$u - b\dot{x} = M\ddot{x}$$

Take Laplace transform

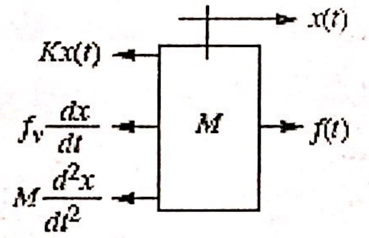
$$U(s) - bs\,X(s) = Ms^2 X(s)$$

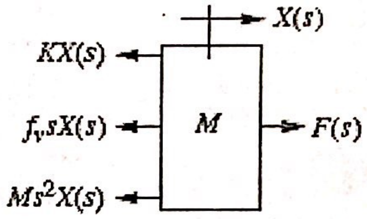
$$\frac{X(s)}{U(s)} = \frac{1}{Ms^2 + bs}$$

Ex: The figure below shows a simple mechanical translational system with a mass, a spring, and a damper. A force F is applied to the system. Derive a transfer function and step responce for the system.



Begin the solution by drawing the free-body diagram shown in Figure below. We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it.





We now write the differential equation of motion using Newton's law to sum to zero all of the forces shown on the mass in the figure.

$$M\frac{d^2x(t)}{dt^2}+f_v\frac{dx(t)}{dt}+Kx(t)=f(t)$$

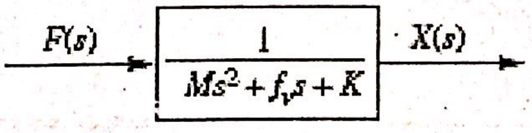
Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s)+f_vsX(s)+KX(s)=F(s)$$

or

$$(Ms^2+f_vs+K)X(s)=F(s)$$

$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+f_vs+K}$$

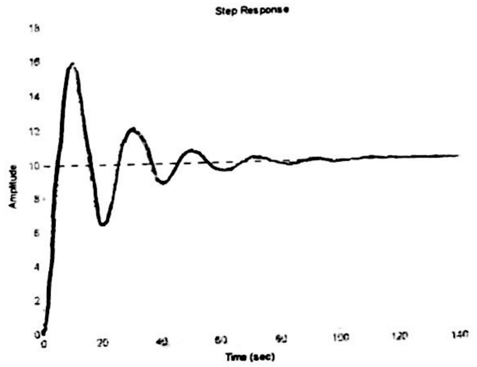


Note: [some of impedance]X(s) = [sum of applied force]

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

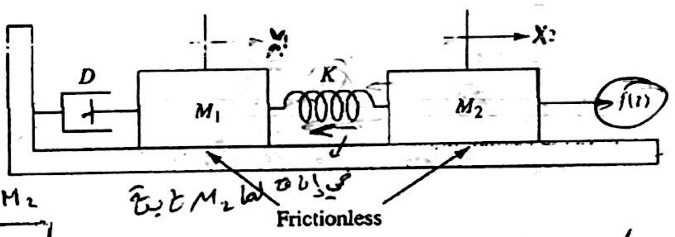
$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

```
% Force is increased in unit step
M=1.0; % kg
fv=0.1;
K=0.1;
Num = [1];
Den=[M fv K];
Step (Num, Den)
```



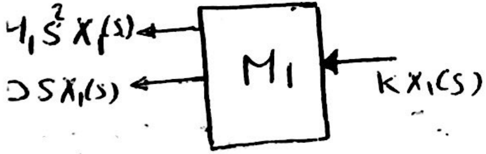
EX: Find the transfer function, G(s) = X2(s)/F(s) for the translational mechanical network shown in Figure below

أول مرة نثبت M_2 ونحرك M_1

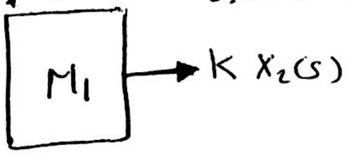


في إزاحة لما M_2 تتحرك

Forces on M_1 due only to motion of M_1



Forces on M_1 due only to motion of M_2

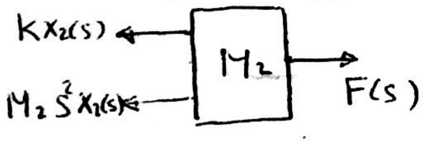


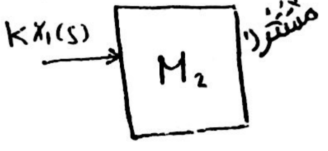
( ) x_1 - ( ) x_2

$$M_1 s^2 X_1(s) + DsX_1(s) + KX_1(s) - KX_2(s) = 0$$

$$\boxed{(M_1 s^2 + Ds + K)X_1(s) - \underline{K} \cdot X_2(s) = 0}$$

مشترك





$$M_2 s^2 X_2(s) + KX_2(s) - KX_1(s) = F(s)$$

$$\therefore \boxed{(M_2 s^2 + K)X_2(s) - \underline{KX_1(s)} = F(s)}$$

العامل المشترك بينهم

$$X_2(s) = \frac{\begin{vmatrix} M_1 s^2 + Ds + K & 0 \\ -K & F(s) \end{vmatrix}}{\begin{vmatrix} M_1 s^2 + Ds + K & -K \\ -K & (M_2 s^2 + K) \end{vmatrix}}$$

$$\Rightarrow \frac{X_2(s)}{F(s)} = \frac{num}{den}$$

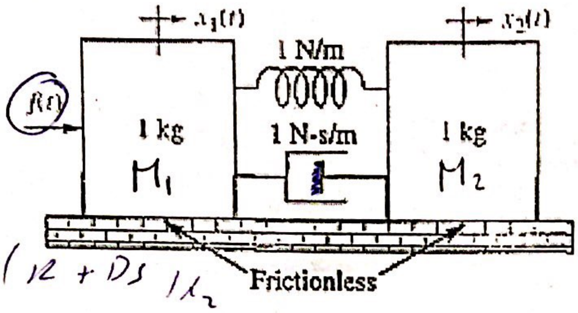
Two mass- system

EX: Find the transfer function, G(s) = X2(s)/F(s) for the translational mechanical network shown in Figure below

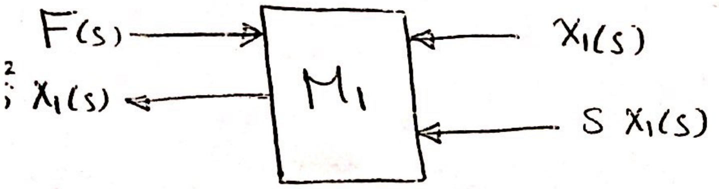
$K = 1$ N/m

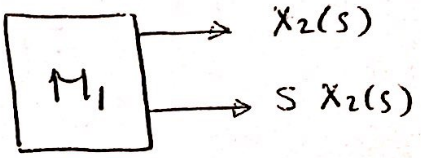
$f_v = 1$ N.s/m

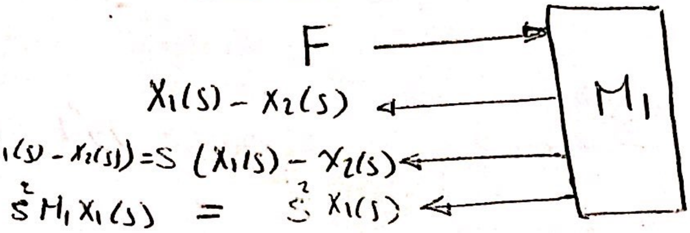
$\left(x_1\right)$ القوة مطبقة على ←



$(Ms_2 + K + Ds)x_1 - (K + Ds)x_2$



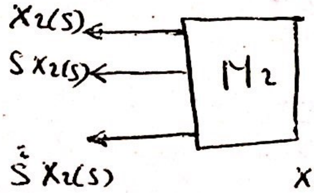


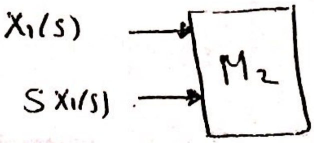


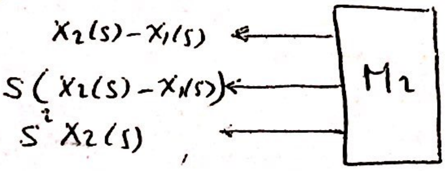
F
M_1
$X_1(s) - X_2(s)$
$f_v(X_1(s) - X_2(s)) = s(X_1(s) - X_2(s))$
$s^2 M_1 X_1(s) = s^2 X_1(s)$

$$s^2 X_1(s) + s(X_1(s) - X_2(s)) + (X_1(s) - X_2(s)) = F(s)$$

$$\boxed{(s^2 + s + 1)X_1(s) - (s+1)X_2(s) = F(s)}$$







$$s^2 X_2(s) + s(X_2(s) - X_1(s)) + X_2(s) - X_1(s) = 0$$

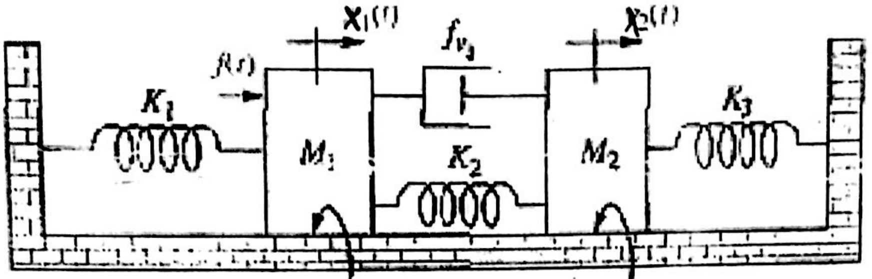
$$\boxed{(s^2 + s + 1)X_2(s) - (s+1)X_1(s) = 0}$$

$$X_2(s) = \frac{\begin{vmatrix} (s^2+s+1) & F(s) \\ -(s+1) & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+s+1) & -(s+1) \\ -(s+1) & (s^2+s+1) \end{vmatrix}}$$

$$\therefore \boxed{\frac{X_2(s)}{\;} = \frac{s+1}{\;}}$$

68

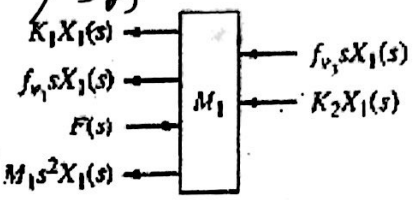
EX: Find the transfer function, X2(s)/F(s), for the system of figure below.



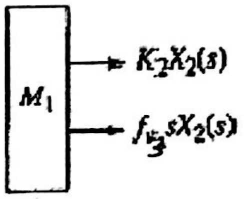
ق احتكاك f_{v_1} f_{v_2}

$(k_1 + Ms^2 + f_v s + k_2 + f_{v_4} s) = F(s)$ $= f_3$

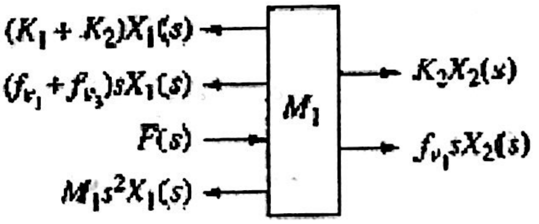
If we **hold M2** still and **move** M1to the right, we see the forces as shown



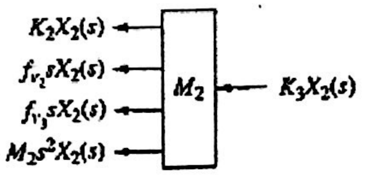
If we hold M1 still and move M2 to the right, we see the forces as shown

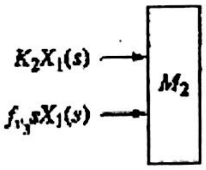


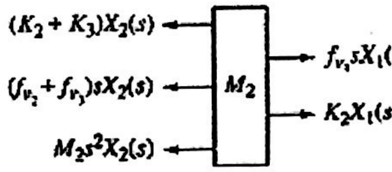
The total force on M1 is the superposition, or sum, of the forces just discussed.



For M2, we proceed in a similar fashion:







The Laplace transform of the equations of motion can now be written from Figures

$$[M_1s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - (f_{v_3}s + K_2)X_2(s) = F(s)$$

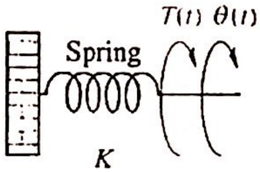
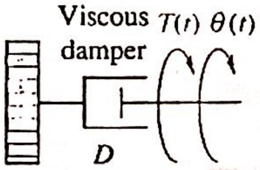
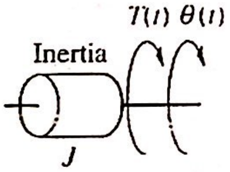
$$-(f_{v_3}s + K_2)X_1(s) + [M_2s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)]X_2(s) = 0$$

⋮ Complete

TF

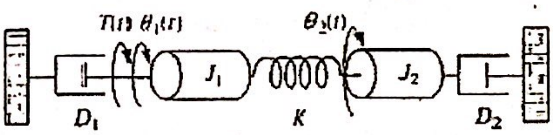
Rotational Mechanical System

Rotational mechanical systems are handled the same way as translational mechanical systems, except that **torque** replaces **force and angular displacement** replaces **translational displacement.** The mechanical components for rotational systems are the same as those for translational systems, except that the components undergo **rotation instead of translation.**

Component	**Torque-angular velocity**	**Torque-angular displacement**	**Impedence** $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	N.m rad $T(t) = K\theta(t)$ Spring Constant	$\frac{N.m}{rad}$ K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$ Coefficient of viscous	$\frac{N.m.s}{rad}$ Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	kg.m² $T(t) = J\frac{d^2\theta(t)}{dt^2}$ Moment of inertia	$\frac{N.m.s^2}{rad}$ Js^2

EX : Find the transfer function, $\theta_2(s)/T(s)$, for the rotational system shown in Figure below

$$\left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_1\end{array}\right]\theta_1(s) - \left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{array}\right]\theta_2(s) = \left[\begin{array}{c}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_1\end{array}\right]$$



$$-\left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{array}\right]\theta_1(s) + \left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_2\end{array}\right]\theta_2(s) = \left[\begin{array}{c}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_2\end{array}\right]$$

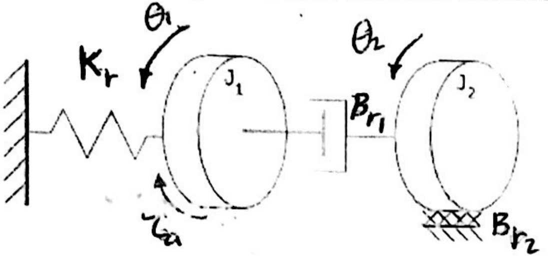
$$(J_1s^2 + D_1s + K)\theta_1(s) \qquad - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2s^2 + D_2s + K)\theta_2(s) = 0$$

$$\theta_2(s) = \frac{\begin{vmatrix} J_1s^2 + D_1s + K & T(s) \\ -K & 0 \end{vmatrix}}{\begin{vmatrix} J_1s^2 + D_1s + K & -K \\ -K & J_2s^2 + D_2s + K \end{vmatrix}}$$

$$\frac{\theta_2(s)}{T(s)} = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

EX: Derive equations of motion for the system and find the transfer function

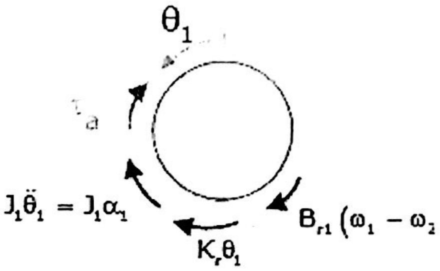


$$\omega = \dot\theta$$

$$\mathcal{L}[\dot\theta] = s\theta(s)$$

$$\mathcal{L}[\ddot\theta] = s^2\theta(s)$$

Free body diagram at θ_1



There are 4 torques acting:

1- The external torque, τ_a, clockwise.

2-The torque due to K_r.
If θ_1 increases (counterclockwise), K_r causes a clockwise torque on J_1.
The resulting torque is $K_r \cdot \theta_1$, clockwise.

3- The torque due to B_{r1}.
If θ_1 increases, the resulting torque on J_1 is $B_{r1} \cdot \omega_1$ in the clockwise direction.
If θ_2 increases, the resulting torque on J_1 is $B_{r1} \cdot \omega_2$ in the counterclockwise direction.
The torque due to B_{r1} is thus $B_{r1} \cdot (\omega_1 - \omega_2)$, clockwise.

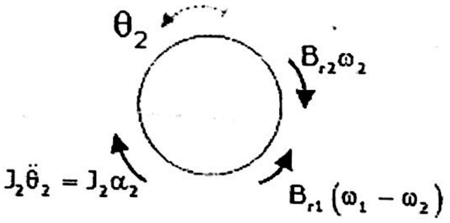
4- The torque due to J_1 (don't forget this - the inertial torque!).
The resulting torque is $J_1 \cdot \alpha_1$ clockwise (the inertial torque is always in the opposite direction from the define positive direction).

$$\tau_a + J_1\alpha_1 + B_{r1}(\omega_1 - \omega_2) + K_r\theta_1 = 0$$

$$J_1\ddot\theta_1 + B_{r1}\dot\theta_1 + K_r\theta_1 - B_{r1}\dot\theta_2 = -\tau_a \longrightarrow ①$$

Free body diagram at θ_2

There are 3 torques acting:



1- The torque due to B_{r2}.
If θ_2 increases (counterclockwise), the resulting torque is $B_{r2} \cdot \omega_2$, clockwise.

2- The torque due to B_{r1}.
If θ_1 increases, the resulting torque on J_2 is $B_{r1} \cdot \omega_1$ in the counterclockwise direction.
If θ_2 increases, the resulting torque on J_2 is $B_{r1} \cdot \omega_2$ in the clockwise direction.
The torque due to B_{r1} is thus $B_{r1} \cdot (\omega_2 - \omega_1)$, clockwise (or $B_{r1} \cdot (\omega_1 - \omega_2)$ counterclockwise).
(Note: this result is trivial because the torque on one end of Br1 must be equal and opposite to the torque on the other end (calculated for the free body diagram for θ_1).)

3- The torque due to J_2 (the inertial torque).
The resulting torque is $J_2 \cdot \alpha_2$ clockwise.

$$J_2\alpha_2 + B_{r2}\omega_2 - B_{r1}(\omega_1 - \omega_2) = 0$$

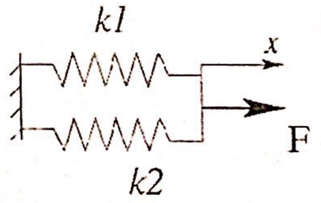
$$J_2\ddot\theta_2 + (B_{r2} + B_{r1})\dot\theta_2 - B_{r1}\dot\theta_1 = 0 \longrightarrow ②$$

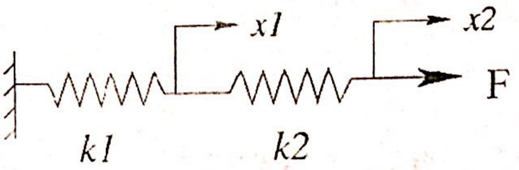
. Takeing the LaPlace transform, assuming zero initial conditions,

$$(J_1 s^2 + B_{r_1} s + k_r)\theta_1(s) - B_{r_1} s\,\theta_2(s) = -T_a(s) \longrightarrow ③$$

9 $(J_2 s^2 + B_{r_2} s + B_{r_1} s)\theta_2(s) - B_{r_1} s\,\theta_1(s) = 0$

EX: Given two springs with spring constant k1 and k2, obtain the equivalent spring constant k_{eq} for the two springs connected in (1) parallel (2) serial.





1- Parallel : The two springs have same displacement, therefore

$$k_1x + k_2x = F = k_{eq}x, \qquad k_{eq} = k_1 + k_2.$$

2- Series : The forces on each spring are same, F. However, their displacements are different. Let them be x1 and x2. Then,

$$k_1x_1 = k_2x_2 = F$$

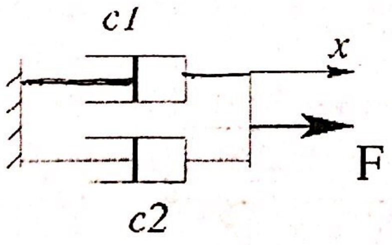
$$x_1 = \frac{F}{k_1} \qquad x_2 = \frac{F}{k_2}$$

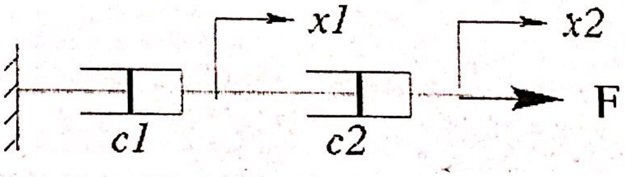
Since the total movement is x = x1 + x2, and we have F = k_{eq} x, then we can obtain

$$\frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2}$$

$$k_{eq} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = \frac{k_1k_2}{k_1 + k_2}.$$

EX: Given two dampers with damping constant c_1 and c_2, obtain the equivalent damping constant c_{eq} for the two dampers connected in (1) parallel (2) serial.:





(1) in parallel $c_{eq} = c_1 + c_2$ (2) in serial $c_{eq} = \frac{c_1c_2}{c_1 + c_2}$

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Skill-Assessment Exercise 2.8- Nise

Find the transfer function, G(s) = X2(s)/F(s), for the translational mechanical system

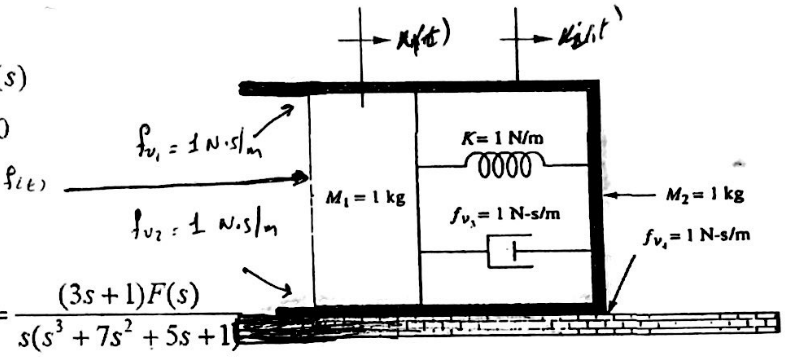
- Writing the equations of motion,

$$(s^2+3s+1)X_1(s)-(3s+1)X_2(s)=F(s)$$

$$-(3s+1)X_1(s)+(s^2+4s+1)X_2(s)=0$$

Solving for $X_2(s)$,

$$X_2(s)=\frac{\begin{vmatrix}(s^2+3s+1) & F(s)\\ -(3s+1) & 0\end{vmatrix}}{\begin{vmatrix}(s^2+3s+1) & -(3s+1)\\ -(3s+1) & (s^2+4s+1)\end{vmatrix}}=\frac{(3s+1)F(s)}{s(s^3+7s^2+5s+1)}$$



- Hence,

$$\frac{X_2(s)}{F(s)}=\frac{(3s+1)}{s(s^3+7s^2+5s+1)}$$

Skill-Assessment Exercise 2.9 - Nise

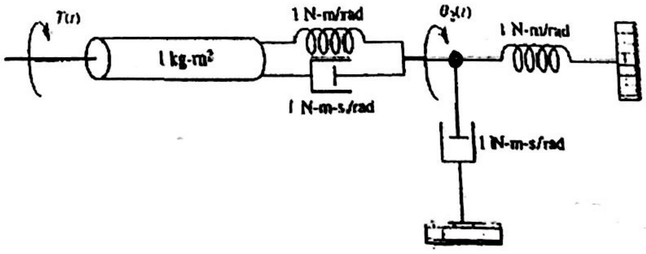
Find the transfer function, G(s) = 2(s)/T(s), for the rotational mechanical system

Writing the equations of motion,

$$(s^2+s+1)\theta_1(s)-(s+1)\theta_2(s)=T(s)$$

$$-(s+1)\theta_1(s)+(2s+2)\theta_2(s)=0$$

where $\theta_1(s)$ is the angular displacement of the inertia.

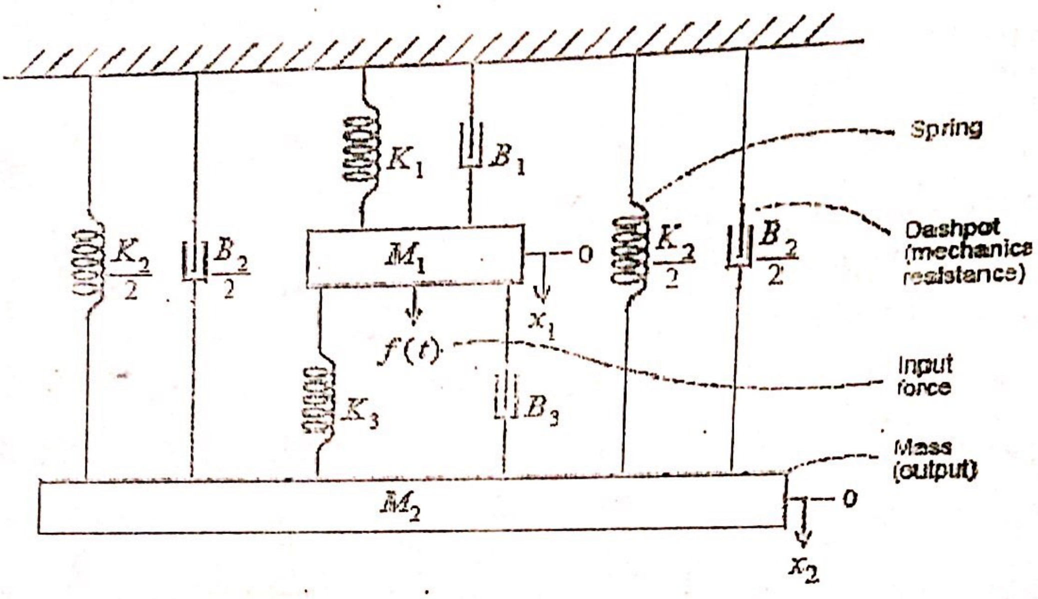


Solving for $\theta_2(s)$,

$$\theta_2(s)=\frac{\begin{vmatrix}(s^2+s+1) & T(s)\\ -(s+1) & 0\end{vmatrix}}{\begin{vmatrix}(s^2+s+1) & -(s+1)\\ -(s+1) & (2s+2)\end{vmatrix}}=\frac{(s+1)F(s)}{2s^3+3s^2+2s+}$$

From which, after simplification,

Ex. Given the mechanical system



Find the transfer function G(s) for the system

$$F(s) \longrightarrow \boxed{G(s)} \longrightarrow X_2(s)$$

LIST THE FORCES ON EACH MASS

Forces on M_1:	$-K_1x_1$ $K_3(x_2-x_1)$ $f(t)$	$-B_1\dot{x}_1$ $B_3(\dot{x}_2-\dot{x}_1)$
Forces on M_2:	$-K_2x_2$ $-K_3(x_2-x_1)$	$-B_2\dot{x}_2$ $-B_3(\dot{x}_2-\dot{x}_1)$

WRITE THE DIFFERENTIAL EQUATIONS FOR EACH MASS

$$M_1\ddot{x}_1 = -K_1x_1 - B_1\dot{x}_1 + K_2(x_2 - x_1) + B_3(\dot{x}_2 - \dot{x}_1) + f(t)$$

$$M_2\ddot{x}_2 = -K_2x_2 - B_2\dot{x}_2 - K_3(x_2 - x_1) - B_3(\dot{x}_2 - \dot{x}_1)$$

CONVERT TO THE S-DOMAIN

$$M_1 s^2 X_1(s) = -K_1 X_1(s) + K_3\left[X_2(s) - X_1(s)\right] - B_1 s X_1(s) + B_3\left[s X_2(s) - s X_1(s)\right] + F(s)$$

$$M_2 s^2 X_2(s) = -K_2 X_2(s) - B_2 s X_2(s) - K_3\left[X_2(s) - X_1(s)\right] - B_3\left[s X_2(s) - s X_1(s)\right]$$

REARRANGE AS SUMS OF THE NET FORCES

$$M_1 s^2 X_1(s) + K_1 X_1(s) - K_3 X_2(s) - K_3 X_1(s) + B_1 s X_1(s) - B_3 s X_2(s) + B_3 s X_1(s) = F(s)$$

$$M_2 s^2 X_2(s) + K_2 X_2(s) + B_2 s X_2(s) + K_3 X_2(s) - K_3 X_1(s) + B_3 s X_2(s) - B_3 s X_1(s) = 0$$

PLACE IN MATRIX FORM

$$\begin{array}{ll} & \begin{array}{cc} X_1 \text{ terms:} & X_2 \text{ terms:} \end{array} \\ \begin{array}{l} M_1 \text{ terms:} \\ M_2 \text{ terms:} \end{array} & \begin{bmatrix} M_1 s^2 + K_1 + B_1 s + K_3 + B_3 s & -K_3 - B_3 s \\ -K_3 - B_3 s & M_2 s^2 + K_2 + B_2 s + K_3 + B_3 s \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix} \end{array}$$

SOLVE FOR $X_2(s)$

$$X_2(s) = \frac{\begin{vmatrix} M_1 s^2 + K_1 + B_1 s + K_3 + B_3 s & F(s) \\ -K_3 - B_3 s & 0 \end{vmatrix}}{\begin{vmatrix} M_1 s^2 + K_1 + B_1 s + K_3 + B_3 s & -K_3 \quad B_3 s \\ -K_3 - B_3 s & M_2 s^2 + K_2 + B_2 s + K_3 + B_3 s \end{vmatrix}}$$

$$X_2(s) = \frac{-F(s)\left[-K_3 -- B_3 s\right]}{\left[M_1 s^2 + K_1 + B_1 s + K_3 + B_3 s\right]\left[M_2 s^2 + K_2 + B_2 s + K_3 + B_3 s\right] - \left(-K_3 - B_3 s\right)^2}$$

SOLVE FOR $G(s)$

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{K_3 + B_3 s}{\left[M_1 s^2 + K_1 + B_1 s + K_3 + B_3 s\right]\left[M_2 s^2 + K_2 + B_2 s + K_3 + B_3 s\right] - \left(-K_3 - B_3 s\right)^2}$$

Home Work 2 ------------------EE362

Problem 1

What is the solution of the following differential equation?

$$2\ddot{x} + 7\dot{x} + 3x = 0, \qquad x(0) = 3, \qquad \dot{x}(0) = 0$$

Repeat the solution using MATLAB

Problem 2

Obtain the partial-fraction expansion of the following function with MATLAB:

$$F(s) = \frac{10(s+2)(s+4)}{(s+1)(s+3)(s+5)^2}$$

Then, obtain the inverse Laplace transform of $F(s)$.

Problem 3

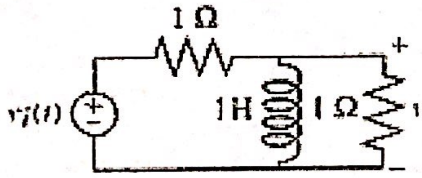
Use MATLABand the symbolic Math Toolbox to find the inverse Lablace Transform.

a. $G(s) = \dfrac{(s^2+3s+10)(s+5)}{(s+3)(s+4)(s^2+2s+100)}$

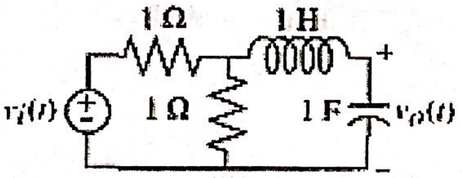
b. $G(s) = \dfrac{s^3+4s^2+2s+6}{(s+8)(s^2+8s+3)(s^2+5s+7)}$

Problem 4

Find the transfer function, $G(s) = V_o(s)/V_i(s)$, for each network (a) ; (b)



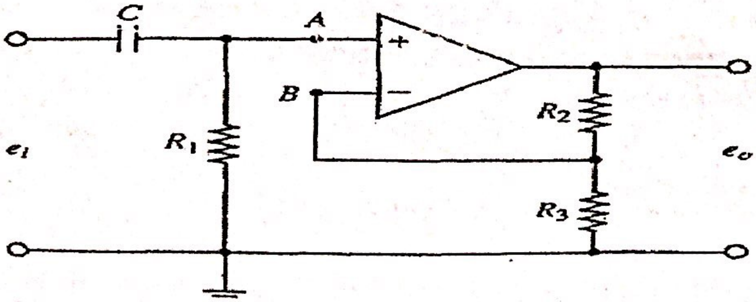
(a)



(b)

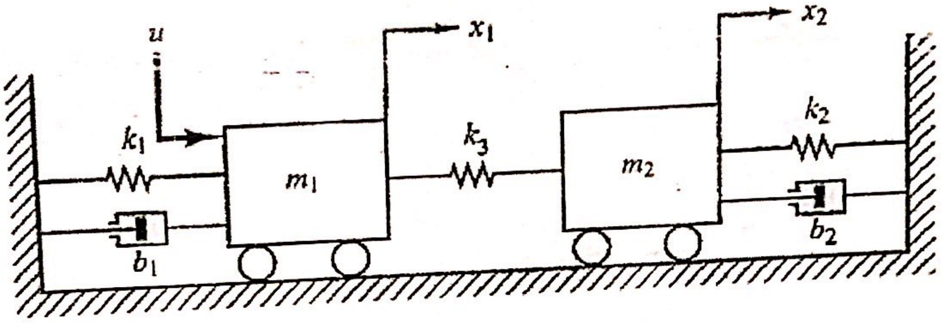
Problem 5

Obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp circuit



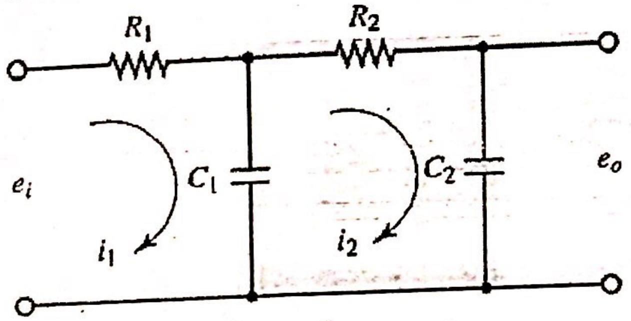
Prpblem 6

Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$ of the mechanical system.



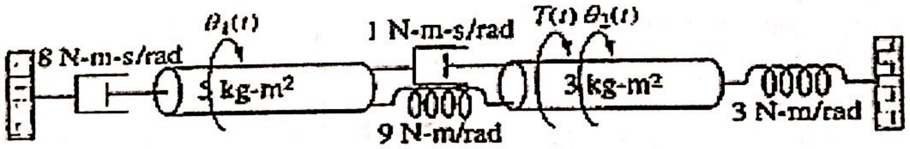
Problem 7

Consider the electrical circuit·Obtain the transfer function $E_o(s)/E_i(s)$

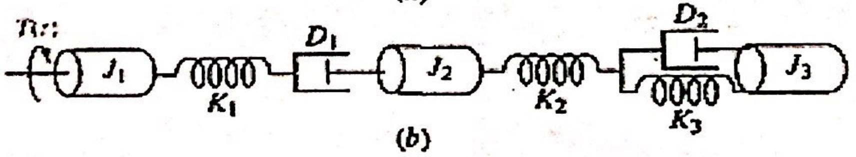


Problem 8

For each of the rotational mechanical systems (a and b) Find the transfer function 2/T(s)



(a)



(b)